Faculty Working Papers

SIMILARITIES UNDERLYING ACCOUNTING APPLICATIONS OF MATRIX ALGEBRA

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In recent years matrix algebra has been applied to a large number of accounting problems. Among the applications most widely discussed are cost allocation and estimating the allowance for doubtful accounts. Unfortunately, because of the way these and other applications of matrix algebra are formulated in the literature, they appear fundamentally different and unrelated. As a consequence, students may memorize individual applications of matrix algebra in a time consuming manner and never understand the underlying concepts well enough to apply them to new situations.

The purpose of this paper is to demonstrate how the applications of matrix algebra mentioned above can be formulated and examined in a manner that brings out their similarities. When there is a conflict between clarity of presentation and computational efficiency, clarity of presentation is emphasized. The author has found from experience that this approach reduces the class time required to cover these and other applications of matrix algebra.

[illegible]

The cost allocation [illegible] allocating service department costs to production departments when the service departments provide reciprocal services to each other. When this condition exists the direct and step allocation methods are found wanting because they do not recognize the reciprocal relations between service departments. Matrix algebra provides at least two methods for solving this problem. First, the direct costs of each production and service department can be formulated as a

series of linear equations and the total costs of the production departments found in a manner similar to that presented in finite mathematics courses. Second, the relationships between each department can be formulated as a transition probability matrix and the portion of each service department's costs that are ultimately allocated to each production department can be found by appropriate matrix operations.

Linear Algebra:

In linear algebra, the solution to a system of n linear equations with n unknowns can be found by first placing the equations in matrix notation and then postmultiplying the inverse of the coefficient matrix by the vector of knowns. The solution to:

$$AX = B \tag{1}$$

is:

$$X = A^{-1}B \tag{2}$$

where:

X = vector of unknowns;

A = coefficient matrix; and

B = vector of knowns.

This approach can be easily applied to the situation presented by Churchill in an early article on cost allocation.[1] There are three service departments, S_1, S_2, and S_3 with direct costs of \$2,000; \$2,000; and \$5,000 respectively, and four production departments, A, B, C, and D with direct costs of \$10,000; \$12,000; \$14,000; and \$8,000. The service departments allocate their costs as follows:

From \ To	A	B	C	D	S_1	S_2	S_3
S_1	.2	.4	.1	.1	0	0	.2
S_2	.1	.2	0	.2	.2	0	.3
S_3	.1	.1	.3	.4	0	.1	0

The direct costs of each department can be expressed with the following series of linear equations:

$$\begin{aligned}
&1A-0B-0C-0D-.2S_1-.1S_2-.1S_3 = 10{,}000\\
&-0A+1B-0C-0D-.4S_1-.2S_2-.1S_3 = 12{,}000\\
&-0A-0B+1C-0D-.1S_1-0S_2-.3S_3 = 14{,}000\\
&-0A-0B-0C+1D-.1S_1-.2S_2-.4S_3 = 8{,}000\\
&-0A-0B-0C-0D+1S_1-.2S_2-0S_3 = 2{,}000\\
&-0A-0B-0C-0D-0S_1+1S_2-.1S_3 = 2{,}000\\
&-0A-0B-0C-0D-.2S_1-.3S_2+1S_3 = 5{,}000
\end{aligned} \qquad (3)$$

It should be noted that A, B, C, D, S_1, S_2, and S_3 represent the total costs that flow to or through these departments, not the direct costs of these departments. The direct costs of department A are equal to the variable A less the costs transferred in from departments S_1, S_2, and S_3. Failure to understand the distinction between the direct costs of a department and the total costs that flow to or through a department can lead to confusion.[2]

In matrix notation the above system of linear equations is expressed as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -.2 & -.1 & -.1 \\ 0 & 1 & 0 & 0 & -.4 & -.2 & -.1 \\ 0 & 0 & 1 & 0 & -.1 & 0 & -.3 \\ 0 & 0 & 0 & 1 & -.1 & -.2 & -.4 \\ 0 & 0 & 0 & 0 & 1 & -.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -.1 \\ 0 & 0 & 0 & 0 & -.2 & -.3 & 1 \end{bmatrix} * \begin{bmatrix} A \\ B \\ C \\ D \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 12{,}000 \\ 14{,}000 \\ 8{,}000 \\ 2{,}000 \\ 2{,}000 \\ 5{,}000 \end{bmatrix} \qquad (4)$$

The total costs of the production departments can now be determined by multiplying the vector of direct costs and the inverse of the coefficient matrix.

$$\begin{bmatrix} A \\ B \\ C \\ D \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & [illegible] & [illegible] & .01801 \\ 0 & 1 & 0 & 0 & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 1 & 0 & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 0 & 1 & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 0 & 0 & [illegible] & [illegible] & .02070 \\ 0 & 0 & 0 & 0 & .02070 & [illegible] & [illegible] \\ 0 & 0 & 0 & 0 & [illegible] & .35199 & 1.03520 \end{bmatrix} \cdot \begin{bmatrix} 10,000 \\ 12,000 \\ 14,000 \\ 8,000 \\ 2,000 \\ 2,000 \\ 5,000 \end{bmatrix} = \begin{bmatrix} 11,398 \\ 14,166 \\ 16,141 \\ 11,296 \\ 2,526 \\ 2,629 \\ 6,294 \end{bmatrix} \qquad (5)$$

Hence, the total costs in production departments A, B, C, and D are \$11,398; \$14,166; \$16,141; and \$11,296 respectively. These amounts total to \$53,001, the sum of the direct costs of the production and service departments.[3] An immediate advantage of this presentation is its similarity to the solution of any system of linear equations. It is strange that the solution to the cost allocation problem is not presented this way in the literature.[4]

It should be noted that the costs that flow through the service departments exceed the direct costs of those departments because costs are reallocated back and forth between them a number of times before they are finally allocated to, or absorbed by, the production departments. In reality, an absorbing Markov process is taking place. Accordingly, the cost allocation problem can also be formulated as an absorbing Markov process.

Markov Process:

The transition probability matrix for an absorbing Markov process has the following standard from:[5]

$$\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right) \qquad (6)$$

where:

> I = probability of going from one absorbing state to another (identity matrix);
>
> O = probability of going from an absorbing state to a nonabsorbing state (zero matrix);
>
> R = probability of going from a nonabsorbing state to an absorbing state; and
>
> Q = probability of going from one nonabsorbing state to another.

Ultimately, the process will be absorbed. The important questions are how many transitions will it take for the process to be absorbed and in what absorbing state will values in the nonabsorbing states end up. The first question is answered by solving the following equation:

$$N = (I-Q)^{-1} \tag{7}$$

where:

> N = average number of times a value in a nonabsorbing state will be in various nonabsorbing states before it is absorbed.

The second question is answered by solving:

$$B = N \cdot R \tag{8}$$

where:

> B = portion of the value in a nonabsorbing state that will end up in various absorbing states.

Once B is determined, the [illegible] of the direct costs of various service departments can be found by postmultiplying the row vector of direct service department costs by B.

Once again, consider the situation presented by Churchill. The transition probability matrix for production and service department costs is as follows:

$$\begin{array}{c|ccccccc}
\text{From} \backslash \text{To} & A & B & C & D & S_1 & S_2 & S_3 \\
\hline
A & 1 & 0 & 0 & & 0 & 0 & 0 \\
B & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
S_1 & .2 & .4 & .1 & .1 & 0 & 0 & .2 \\
S_2 & .1 & .2 & 0 & .2 & .2 & 0 & .3 \\
S_3 & .1 & .1 & .3 & .4 & 0 & .1 & 0
\end{array} \qquad (9)$$

The matrix indicates the disposition of costs that flow through various departments during one iteration of the allocation process. For example, of the costs that flow through service department S_1, 20 percent go to production department A, 40 percent go to department B, etc.

To find the average number of times a direct service department cost flows through various service departments before it is absorbed, solve for N.

$$N = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & .2 \\ .2 & 0 & .3 \\ 0 & .1 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1.00414 & 0.02070 & 0.20704 \\ 0.20704 & 1.03520 & 0.35199 \\ 0.02070 & 0.10350 & 1.03520 \end{bmatrix} \qquad (10)$$

For example, in the allocation process a dollar of direct costs in S_1 will flow through S_1 an average of 1.00414 times, S_2 an average of 0.02070 times, and S_3 an average of 0.20704 times before it is absorbed. The similarity of the numbers in (10) and those in the lower right hand corner of (5) should be noted.

The total flow of dollars through the service departments is easily found by multiplying (10) and the vector of direct service departments costs:

$$\begin{bmatrix} 2,000 & 2,000 & 5,000 \end{bmatrix} \cdot \begin{bmatrix} 1.00414 & 0.02070 & 0.20704 \\ 0.20704 & 1.03520 & 0.35199 \\ 0.02070 & 0.10350 & 1.03520 \end{bmatrix} =$$

$$\begin{bmatrix} 2,526 & 2,629 & 6,294 \end{bmatrix} \qquad (11)$$

As expected the [illegible] flow of costs through the service departments exceeds the direct costs of the service departments. This answer can be compared with that obtained in (5).

To find the portion of the dollars in various service departments that will ultimately end up in a particular production department, solve for B.

$$B = \begin{bmatrix} 1.00414 & 0.02070 & 0.20704 \\ 0.20704 & 1.03520 & 0.35199 \\ 0.02070 & 0.10350 & 1.03520 \end{bmatrix} \cdot \begin{bmatrix} .2 & .4 & .1 & .1 \\ .1 & .2 & .0 & .2 \\ .1 & .1 & .3 & .4 \end{bmatrix} =$$

$$\begin{bmatrix} 0.22360 & 0.42650 & 0.16253 & 0.18737 \\ 0.18013 & 0.32506 & 0.12630 & 0.36854 \\ 0.11801 & 0.13250 & 0.31263 & 0.43685 \end{bmatrix} \tag{12}$$

For example, a dollar of direct costs in S_1 will ultimately be allocated to A, B, C, and D in accordance with values in the first row of (12). The similarity of the numbers in (12) and those in the upper right had corner of (5) should be noted.

The ultimate allocation of direct service department costs can be found by multiplying (12) and the row vector of direct service department costs:

$$\begin{bmatrix} 2,000 & 2,000 & [illegible] \end{bmatrix} \cdot \begin{bmatrix} 0.22360 & 0.42650 & 0.16253 & 0.18737 \\ [illegible] & [illegible] & 0.12630 & 0.36854 \\ [illegible] & [illegible] & 0.31263 & 0.43685 \end{bmatrix} =$$

$$\begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] \end{bmatrix} \tag{13}$$

As expected, when the direct production department costs are added to those allocated from the service departments, the final solution is the same as that obtained when the problem was formulated as a series of linear equations. An advantage of presenting the solution to the cost allocation problem as a series of linear equations and then as an absorbing Markov process is the progression from a procedure to which the student has had previous exposure to a less familiar one.

UNCOLLECTABLE ACCOUNTS ESTIMATION

Once the cost allocation problem has been solved by the use of an absorbing Markov process, estimating the allowance for doubtful accounts becomes merely another application of a previously used concept. Consider the situation presented by Cyert, Davidson, and Thompson in the appendix to their article on doubtful account estimation.[6] There are two absorbing states, $\bar{0}$, an account is collected, and 2, an account is declared bad, and two nonabsorbing states, 0, an account is current, and 1, an account is one period old. The transition probability matrix for movement between these various states is as follows.

$$\begin{array}{c|cccc} \text{From} \setminus \text{To} & \bar{0} & 2 & 0 & 1 \\ \hline \bar{0} & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & .3 & 0 & .5 & .2 \\ 1 & .5 & .1 & .3 & .1 \end{array} \tag{14}$$

To find the average number of times an account is in various non-absorbing states before it is absorbed, solve for N.

$$N = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & .2 \\ .3 & .1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2.31 & .51 \\ .77 & 1.28 \end{bmatrix} \tag{15}$$

For example, an account in state 0 will spend an average of 2.31 periods in state 0 and .51 periods in state 1 before it is absorbed.

To find the portion of the accounts in a nonabsorbing state that will ultimately end up in a particular absorbing state solve for B.

$$B = \begin{bmatrix} 2.31 & .51 \\ .77 & 1.28 \end{bmatrix} \cdot \begin{bmatrix} .3 & 0 \\ .5 & .1 \end{bmatrix} = \begin{bmatrix} .95 & .05 \\ .87 & .13 \end{bmatrix} \tag{16}$$

For example, 95 percent of the accounts in state 0 will ultimately end up in absorbing state $\bar{0}$ and 5 percent of the accounts in state 0 will ultimately end up in absorbing state 2.

If all of the accounts in the nonabsorbing states are of approximately equal size an estimate of the dollar amount of the accounts that will ultimately be collected or go bad can be made. Assume there are \$10,000 in state 0 and \$5,000 in state 1. Then, the expected final disposition of these dollars is as follows:

$$\begin{bmatrix} 10,000 & 5,000 \end{bmatrix} \cdot \begin{bmatrix} .95 & .05 \\ .87 & .13 \end{bmatrix} = \begin{bmatrix} 13,850 & 900 \end{bmatrix} \tag{17}$$

The allowance for uncollectable accounts should be \$900.

SIMILARITIES

Many additional accounting applications of matrix algebra, such as inventory valuation in process costing[7] and consolidated income determination with intercorporate stockholdings[8] may be formulated as either a system of linear equations or a Markov process. Yet, regardless of the way they are formulated the notions of "<u>flow</u>" and "<u>absorption</u>" underly all applications mentioned in this paper.

In cost allocation problems, costs flow through service departments and are absorbed by production departments. In accounts receivable problems, revenues flow through various age categories and are absorbed by being collected or written off. In process costing problems, costs flow through production departments and are absorbed by inventories or the scrap heap. In consolidated income problems with intercorporate stockholdings, income flows through the affiliated corporations and is absorbed by the majority and minority interests.

Once the significance of the notions of "flow" and "absorbtion" is grasped, the student is able to understand the similarities underlying many accounting applications of matrix algebra and solve these problems without the aid of the instrument of a specific example.

CONCLUSIONS

The student's understanding of various accounting applications of matrix algebra can be enhanced if these applications are related to previously learned concepts, such as the solution of a system of linear equations, and if the similarities underlying these applications are emphasized. In so doing a certain amount of computational efficiency may be lost. However, this loss of computational efficiency is not critical in an undergraduate accounting class because of the widespread availability of computers and the size of the problems considered. Just as important as the increased clarity of presentation is the reduction in class time that must be devoted to matrix algebra. The vast increase in the number of issues that should be considered in the classroom requires that each of them be presented more clearly and in a less time consuming manner.

FOOTNOTES

[1]N. Churchill, "Linear Algebra and Cost Allocations: Some Examples," The Accounting Review (October, 1964), pp 894-904.

[2]R. Manes, "Comment on Matrix Theory and Cost Allocation," The Accounting Review (July, 1965), pp. 640-64[illegible]; J. Livingstone, "Matrix Algebra and Cost Allocation," The Accounting Review (July, 1968), pp. 503-508.

[3]There is a one dollar rounding error in the calculations.

[4]Dispite the fact that a system of linear equations is normally solved in matrix algebra by multiplying the vector of constants by the inverse of the coefficient matrix, in cost allocation models, most authors break the coefficient matrix down into a number of smaller matricies and perform additional operations on them. While such a procedure may have computational advantages, it lacks clarity of presentation.

[5]J. Kemeny, A. Schleifer, Jr., J. Snell, and G. Thompson, Finite Mathematics, (Prentice Hall, 1972), pp. 224-229.

[6]R. M. Cyert, H. J. Davidson, and G. L. Thompson, "Estimation of the Allowance for Doubtful Accounts by Markov Chains," Management Science (April, 1962), pp. 287-[illegible].

[7]Churchill, pp. 897-900.

[8]C. H. Griffin, T. H. Williams & D. Larson, Advanced Accounting, (Irwin, 1971), pp. 516-519.